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ON EXISTENCE AND COMPLETENESS OF CONSERVATION LAWS ASSOCIATED WITH ELEMENTARY BEAM THEORY

R. KIENZLER
Fraunhofer-Institut für Werkstoffmechanik, Wöhlerstrasse 11, 7800 Freiburg, Federal Republic of Germany

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Abstract—From a variational principle with varying boundaries conservation laws in the context of elementary beam theory are derived. Necessary and sufficient conditions are given for transformations leading to conservation laws. The formulas obtained are applied to several examples.

1. INTRODUCTION

Elementary beam theory made its departure from early works of Bernoulli (1694)[1] and was fully developed in 1744 by Euler[2]. It took about a century until Navier (1821) and Lamé (1828, 1852) discovered the general two- and three-dimensional equations of linear elastostatics.

Considerations of the energy momentum tensor or material momentum tensor were stimulated by Eshelby's famous paper (1951)[3]. Since that time several papers were published dealing with two- and three-dimensional equations of material conservation and balance laws. Surprisingly, however, no paper has been published until now (to the author's knowledge) that is concerned with material conservation laws in a one-dimensional description, i.e. conservation laws based on a strength-of-materials type of analysis.

Different possibilities exist to obtain conservation laws in Euler–Bernoulli beam theory. Whereas in Ref. [4] a virtual work theorem is considered, this paper uses a more general (and very formal) approach. A variational principle with varying boundaries is established to deliver physical as well as material conservation and balance laws. Roughly speaking, transformations are investigated which leave the action integral unchanged or invariant. In addition to the classical (physical) conservation laws (i.e. conservation of force and conservation of moment), two new (material) conservation laws are obtained. A reflection on necessary and sufficient conditions for action insensitive transformations proves that the set of conservation laws is complete.

Application of the formulas is illustrated with several examples.

2. GOVERNING EQUATIONS

Consider a beam element of length dx (Fig. 1). All quantities involved are functions of the independent variable x. Derivation with respect to x is marked by a dash. q and m are the given (conservative) load and moment per unit length, respectively. They are connected with the transverse shear forces Q and the bending moment M by the equations of equilibrium

$$Q' = -q, \tag{1a}$$

$$M' = Q - m. \tag{1b}$$

The "constitutive equation" couples M with the bending stiffness EI and the rotation of the cross-section ψ

$$\psi' = \frac{M}{EI}, \tag{1c}$$

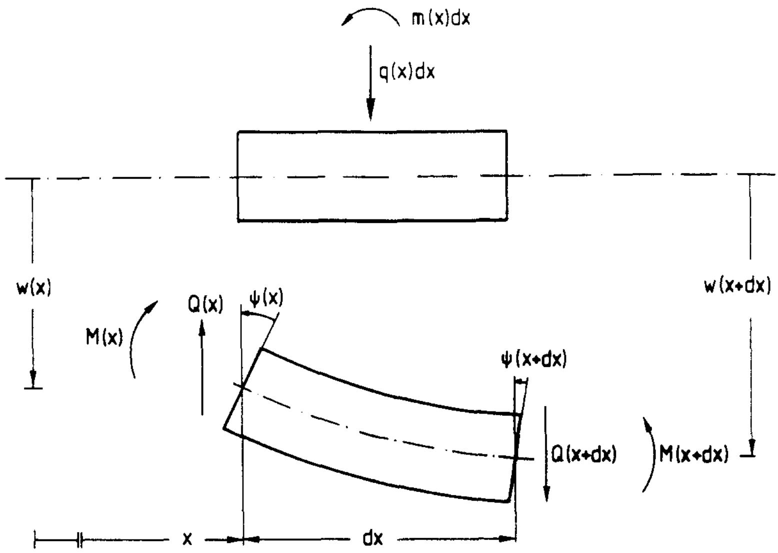

Fig. 1. Beam element.

and the celebrated normal hypothesis states that ψ is equal to the slope of the deflected beam with displacement w

$$w' = -\psi. \tag{1d}$$

The strain energy W and the potential of the external forces V per unit length are given by

$$W = \tfrac{1}{2}EI\psi'^2 = W(x, \psi'), \tag{1e}$$

$$V = -qw - m\psi = V(x, w, \psi). \tag{1f}$$

W and V are explicit functions of x if the stiffness EI and the external forces, q and m, are functions of x. The action integral A associated with a beam of length l is calculated from

$$A = \int_0^l (W+V)\,\mathrm{d}x = \int_0^l L(x, w, \psi, \psi')\,\mathrm{d}x. \tag{2}$$

$W+V$ is the Lagrange function of the problem under consideration. In the next paragraph we will study possible conservation laws following from transformations performed on dependent and independent variables.

3. VARIATIONAL PRINCIPLE WITH VARYING BOUNDARIES

Classical variational principles are concerned with "shape variations" $\bar{\delta}$ of the dependent variable w, e.g.

$$\bar{\delta}w = w^*(x) - w(x).$$

It is important to note that the arguments of w^* and w are identical. In order to derive physical and material conservation laws the class of variations is too restricted. In addition to the local variation "convective variations", related to the variation of the independent variable x, have to be admitted. This leads to a variational principle with varying boundaries[5, 6]. The transformations in the following are supposed to be infinitesimal.

Consider a neighbouring state of the action A

$$A^* = \int_0^l L^*\left(x^*, w^*, \psi^*, \frac{d\psi^*}{dx^*}\right) dx^*, \tag{3}$$

the variation of x is denoted by

$$\delta x = x^* - x. \tag{4a}$$

Proceeding as in Ref. [7], we get the following expressions for the total variations of w, ψ and L

$$\delta w = w^*(x^*) - w(x) = \bar{\delta} w + w'\delta x, \tag{4b}$$

$$\delta\psi = \psi^*(x^*) - \psi(x) = \bar{\delta}\psi + \psi'\delta x, \tag{4c}$$

$$\delta L = L^* - L = \bar{\delta} L + L'\delta x. \tag{4d}$$

The infinitesimal length of the element is transformed to

$$dx^* = (1 + \delta x')\, dx. \tag{4e}$$

It is obvious from eqn (4) that total variation and differentiation are not commutative, whereas local variation and differentiation are, e.g.

$$\delta(w') \neq (\delta w)', \tag{5a}$$

$$\bar{\delta}(w') = (\bar{\delta} w)'. \tag{5b}$$

From (2), (3) and (4d, e) the total variation of the action is

$$\delta A = A^* - A = \int_0^l [\bar{\delta} L + (L\delta x)']\, dx. \tag{6}$$

Proceeding further, the local variation of the Lagrangian, $\bar{\delta} L$, has to be determined

$$\bar{\delta} L = \bar{\delta}(W(x, \psi') + V(x, w, \psi)) = \frac{\partial W}{\partial \psi'}\bar{\delta}\psi' + \frac{\partial V}{\partial w}\bar{\delta} w + \frac{\partial V}{\partial \psi}\bar{\delta}\psi.$$

With (1c–f) it follows

$$\bar{\delta} L = M\bar{\delta}\psi' - q\bar{\delta} w - m\bar{\delta}\psi.$$

Differentiation by parts and frequent use of (1) and (5b) leads to

$$\bar{\delta} L = (M\bar{\delta}\psi)' + (Q\bar{\delta} w)'. \tag{7}$$

Therefore the integrand of (6) can be represented as a complete differential

$$\delta A = \int_0^l (M\bar{\delta}\psi + Q\bar{\delta} w + L\delta x)'\, dx.$$

Using (4b, c) and the abbreviation

$$B = L - M\psi' - Qw', \tag{8}$$

it follows from (6)

$$\delta A = \int_0^l [B\delta x + M\delta\psi + Q\delta w]' \, \mathrm{d}x.$$

Assuming the integrand to be continuous, this integral vanishes if and only if the integrand is zero

$$\delta A = 0 \Leftrightarrow [B\delta x + M\delta\psi + Q\delta w]' = 0. \tag{9}$$

This statement is a conservation law. Equation (9) will, in general, not be satisfied identically for arbitrary transformations. The problem now is to find admissible transformations δx, δw and $\delta\psi$ that fulfil eqn (9), i.e. to find transformations leaving the Lagrangian invariant.

4. GROUPS OF ADMISSIBLE TRANSFORMATIONS AND ASSOCIATED CONSERVATION LAWS

In order to find necessary and sufficient conditions for admissible transformations eqn (9) is rearranged. Firstly, from (4c), (1d) and (5b) it is obvious that $\delta\psi$ is not independent of δx and δw:

$$\delta\psi = -\delta(w') = -\delta w' + w'\delta x'. \tag{10}$$

Inserting (10) and (1) into (9) we end up after some algebra with

$$\delta A = 0 \Leftrightarrow M(\tfrac{3}{2}w''\delta x' - \delta w'' + w'\delta x'') - b\delta x - qw\delta x' - q\delta w + m\delta w' = 0, \tag{11}$$

with a new quantity b given by

$$b = q'w + m'\psi - \tfrac{1}{2}EI'\psi'^2 = -\left.\frac{\partial L}{\partial x}\right|_{\text{expl.}}. \tag{12}$$

If the Lagrangian does not depend explicitly on x it follows immediately from (11) that the transformation $\delta x =$ const, $\delta w = 0$ is admissible leading to the conservation law $B' = 0$. Looking, however, for all admissible transformations I try the "ansatz"

$$\begin{aligned} \delta x &= \Phi(x, w, w'), \\ \delta w &= \Psi(x, w, w'). \end{aligned} \tag{13}$$

Differentiation with respect to x and inserting into (11) leads to the rather clumsy expression

$$\begin{aligned} 0 = {} & -b\Phi - q\Psi - qw\left(\frac{\partial\Phi}{\partial x} - \frac{\partial\Phi}{\partial w}\right) + m\left(\frac{\partial\Psi}{\partial x} + \frac{\partial\Psi}{\partial w}\right) \\ & - \frac{\partial^2\Psi}{\partial x^2} - w'\left(2\frac{\partial^2\Psi}{\partial x\,\partial w} - \frac{\partial^2\Phi}{\partial x^2}\right) - w'^2\frac{\partial^2\Psi}{\partial w^2} \\ & \underline{+ w''\left(\frac{3}{2}\frac{\partial\Phi}{\partial x} - \frac{\partial\Psi}{\partial w} + \frac{5}{2}\frac{\partial\Phi}{\partial w} + 2w'\frac{\partial^2\Phi}{\partial x\,\partial w'}\right.} \\ & \underline{\left. - \frac{\partial^2\Psi}{\partial x\,\partial w'} + 2w'^2\frac{\partial^2\Phi}{\partial w\,\partial w'} - 2w'\frac{\partial\Psi}{\partial w\,\partial w'}\right.} \\ & \underline{\left. - qw\frac{\partial\Phi}{\partial w'} + m\frac{\partial\Psi}{\partial w'}\right)} \\ & \underline{+ w''^2\left(w'\frac{\partial^2\Phi}{\partial w'^2} - \frac{\partial^2\Psi}{\partial w'^2} + \frac{3}{2}\frac{\partial\Phi}{\partial w'}\right)} \\ & \underline{+ w'''\left(w'\frac{\partial\Phi}{\partial w'} - \frac{\partial\Psi}{\partial w'}\right)}. \end{aligned} \tag{14}$$

Notice that all quantities in parentheses are functions of x, w, and w' only. In order to render eqn (14) valid the last three parentheses must vanish identically. This is only possible if the following relations hold

$$\begin{aligned} &\frac{\partial \Phi}{\partial w'} = 0, \\ &\frac{\partial \Phi}{\partial w} = 0, \\ &\frac{\partial \Phi}{\partial x} = \frac{2}{3}\frac{\partial \Psi}{\partial w}, \\ &\frac{\partial \Psi}{\partial w'} = 0, \end{aligned} \tag{15}$$

i.e.

$$\begin{aligned} \Phi &= \Phi(x), \\ \Psi &= \Psi(x, w). \end{aligned}$$

The underlined terms, therefore, vanish and (14) reads much simpler

$$0 = -b\Phi - q\Psi + V\frac{\partial \Phi}{\partial x} + m\frac{\partial \Psi}{\partial x} - \frac{\partial^2 \Psi}{\partial x^2} - \tfrac{1}{2}w'\frac{\partial^2 \Psi}{\partial x\,\partial w} - w'^2\frac{\partial^2 \Psi}{\partial w^2}. \tag{16}$$

Equation (16) cannot be fulfilled without posing any restrictions on the generalized external loads q, m and b. Because of (15) it has to be guaranteed that the last three terms in (16) vanish identically, whereas the first terms reveal the restrictions. Following the same argument as above it follows

$$\frac{\partial^2 \Psi}{\partial x^2} = \frac{\partial^2 \Psi}{\partial x\,\partial w} = \frac{\partial^2 \Psi}{\partial w^2} = 0.$$

This yields, together with (15), a set of four and only four independent groups of admissible transformations

$$\begin{aligned} \Psi &= \delta w = a_1 + a_2 x + \tfrac{3}{2}a_4 w, \\ \Phi &= \delta x = a_3 + a_4 x, \\ \delta\psi &= -a_2 + \tfrac{1}{2}a_4\psi, \quad a_i = \text{const.} \end{aligned} \tag{17}$$

Comparison of (9) and (16) leads with (17) to a set of four conservation laws together with the restrictions imposed on the generalized external forces. In order to show the duality between physical and material description two more abbreviations are introduced

$$\begin{aligned} H &= -\tfrac{1}{2}M - \tfrac{3}{2}Qw, \\ h &= +\tfrac{5}{2}qw + \tfrac{5}{2}m\psi = -\tfrac{5}{2}V. \end{aligned} \tag{18}$$

The complete set of conservation laws associated with each individual group of transformations is

$$a_1: \quad Q' = 0 \quad \text{if} \quad q = 0, \tag{19a}$$

$$a_2: \quad (Qx - M)' = 0 \quad \text{if} \quad q = m = 0, \tag{19b}$$

$$a_3: \quad B' = 0 \quad \text{if} \quad b = 0, \tag{19c}$$

$$a_4: \quad (Bx - H)' = 0 \quad \text{if} \quad b = h = 0. \tag{19d}$$

If no restrictions are posed on the loading conditions, the transformation (17) leads to four corresponding balance laws

$$a_1: \quad Q' = -q, \tag{20a}$$

$$a_2: \quad (Qx-M)' = -qx+m, \tag{20b}$$

$$a_3: \quad B' = -b, \tag{20c}$$

$$a_4: \quad (Bx-H)' = -bx+h. \tag{20d}$$

Conservation of force (19a) is the rather trivial statement that the transverse shear force Q is constant if the beam is not subjected to loads q. The moment is conserved (19b) if force and moment loads are not admitted. The "material force" B is constant (19c) if the "material load" b vanishes, i.e. the beam under consideration is supposed to be homogeneous $[EI(x) = EI_0 = \text{const}]$ and loaded by constant physical forces and moments q_0 and m_0, respectively. Conservation of "material moment" (19d) is possible only if, in addition to $b = 0$, the beam is subjected to transverse shear forces and moments at its ends only $(q = m = 0)$.

Further discussions on the physical significance of B and H and the duality of physical and material field equations are given in Ref. [4]. Application of the new introduced conservation laws to beams with discontinuities in stiffness and the connection with the energy release rate used in fracture mechanics will be studied in a paper dealing especially with defective beams[8].

5. SOME EXAMPLES

Günther[7] derived a conservation law due to the translational invariance of plane stress fields. By introducing the assumptions of elementary beam theory he obtained a balance law for beams. His equation (3.6) is rederived by integrating the balance law (20c) and allowing for single or point forces K_i acting at x_i

$$\int_0^l q(x)w'(x)\,\mathrm{d}x + \sum_i K_i w'(x_i) + \tfrac{1}{2}[M(l)\psi'(l) - M(0)\psi'(0)] + [Q(l)w'(l) - Q(0)w'(0)] = 0. \tag{21a}$$

Proceeding similarly with (20d) another equation is set aside to the previous one

$$\begin{aligned} &\frac{3}{2}\int_0^l q(x)w(x)\,\mathrm{d}x - \int_0^l q(x)w'(x)x\,\mathrm{d}x + \frac{3}{2}\sum_i K_i w(x_i) - \sum_i K_i w'(x_i)x_i \\ &\quad -\tfrac{1}{2}M(l)\psi'(l)l + \tfrac{1}{2}M(l)\psi(l) - \tfrac{1}{2}M(0)\psi(0) \\ &\quad -Q(l)w'(l)l + \tfrac{3}{2}Q(l)w(l) - \tfrac{3}{2}Q(0)w(0) = 0. \end{aligned} \tag{21b}$$

Equations (21) will be applied to some simple examples concerned with beams of constant stiffness EI and length l. Consider a cantilever beam, subjected to constant load q_0 (Fig. 2).

Use of (21) leads to

$$(21\text{a}): \quad q_0 w(l) = \frac{1}{2}\frac{M_{(0)}^2}{EI}, \tag{22a}$$

$$(21\text{b}): \quad q_0 w(l) = \frac{5}{2} q_0 \int_0^l \frac{w(x)}{l}\,\mathrm{d}x. \tag{22b}$$

Since the bending moment at $x = 0$ is $M(0) = q_0 l^2/2$ (22a) leads to the well-known formula

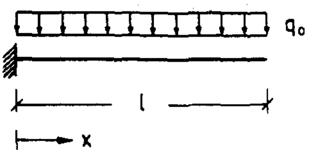

Fig. 2. Cantilever beam under constant load q_0.

to calculate the deflection at the end of the beam

$$w(l) = \frac{q_0 l^4}{8EI}.$$

Equation (22b) states that the end deflection of the beam is 2.5 times the average deflection.

Consider a cantilever beam subjected to a point force K at x_k (Fig. 3). Use of (21) leads to

$$\left.\begin{aligned} &(21\text{a}): && -\frac{1}{2}\frac{M_{(0)}^2}{EI} + Kw'(x_k) = 0 \\ &(21\text{b}): && \frac{3}{2}Kw(x_k) - Kw'(x_k)x_k = 0 \end{aligned}\right\} \quad \begin{aligned} w'(x_k) &= \frac{1}{2}\frac{M_{(0)}^2}{EIK}, \\ w(x_k) &= \frac{1}{3}\frac{M_{(0)}^2 x_k}{EIK}. \end{aligned}$$

Since $M(0) = Kx_k$ two formulas come up to calculate deflection and slope under the point force.

Consider, finally, a simply supported beam subjected to a point force K at x_k (Fig. 4). Use of (21) leads to two formulas which might be helpful in solving statically undetermined beam problems

$$\begin{aligned} &(21\text{a}): && Aw'(0) + Bw'(l) = Kw'(x_k), \\ &(21\text{b}): && Bw'(l)l = Kw'(x_k)x_k - \tfrac{3}{2}Kw(x_k), \end{aligned}$$

with A and B as the supporting forces at $x = 0$ and $x = l$, respectively. The upper equation was given first by Günther[7].

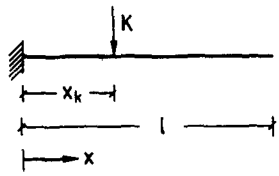

Fig. 3. Cantilever beam under point load K.

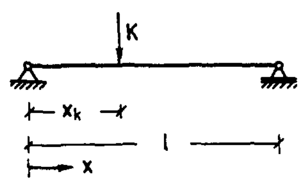

Fig. 4. Simply supported beam under point load K.

6. CONCLUDING REMARKS

Based on a variational principle with varying boundaries a set of four conservation laws is established in the context of Euler–Bernoulli's elementary beam theory. A proof is given on existence and completeness of conservation laws associated with transformations in dependent and independent variables. The new formulas are applied to some illustrative examples.

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ADDENDUM

If the "ansatz" (13) is extended to include the second derivatives of the deflection, w'', an additional transformation is obtained leaving the Lagrangian invariant under the condition that the stiffness, EI, of the beam is constant and load $q = EIw^{IV}$ equals zero

$$\delta x = a_5 w', \qquad \delta w = a_5(w'^2 - \tfrac{1}{2}ww''), \qquad \delta\psi = \tfrac{1}{2}a_5(ww''' - w'w''). \tag{A.1}$$

With the abbreviation

$$R = -\psi^2 - \tfrac{3}{2}w\psi', \tag{A.2}$$

the conservation law associated with transformation (A.1) is given by

$$[Bw' - Hw'' + REIw''']' = 0. \tag{A.3}$$

A similar expression has been given in Ref. [4]

$$[\tfrac{1}{2}Bx^2 - Hx + REI]' = 0. \tag{A.4}$$

For $EIw^{IV} = q = 0$ it is easy to show that (A.3) and (A.4) are equivalent.